

Free Entropy

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Abstract

Free entropy is the analogue of entropy in free probability theory. The paper is a survey of free entropy, its applications to von Neumann algebras, connections to random matrix theory and a discussion of open problems.

0 Introduction

Entropy, from its initial appearance in thermodynamics and passing through statistical mechanics and the mathematical theory of communications of Claude Shannon, has come to play, in various guises, a fundamental role in many parts of mathematics. This article is about a recent addition ([35]) to the mathematical territory of entropy .

Free entropy refers to the analogue of entropy in free probability theory, i.e. a quantity playing the role of entropy in a highly noncommutative probabilistic framework, with independence modelled on free products instead of tensor products. Free probability theory can be viewed as a parallel to some basic probability theory drawn starting from a new type of independence. Surprisingly, the parallelism of the classical and free theories appears to go quite far, as illustrated for instance by the existence of a free entropy theory.

From another perspective, free probability, and in particular free entropy, has deep connections on one hand with the asymptotic behavior of large random matrices and on the other hand with operator algebras. One consequence is that the von Neumann algebras of free groups, once viewed as exotic creatures, are now much better understood and perceived as important objects.

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1 Free Probability Background

1.1 Some basic laws

Free probability theory being a parallel to classical probability theory, we may compare the two by taking a look at corresponding fundamental distributions.

- a) The role of the Gaussian distribution in free probability theory is held by the semi-circle distribution, which is a distribution with compact support.
- b) For the Poisson distribution, the free correspondent is a distribution related to the semi-circle law. It is also a compactly supported distribution which has at most one atom.
- c) The free Cauchy distribution is the Cauchy distribution itself, i.e. the free correspondent is the same as the classical law.

The semi-circle distribution occurs in random matrix theory, where Wigner discovered that it is the limit distribution of eigenvalues of large hermitian Gaussian matrices. Similarly, the free Poisson laws also occur in random matrix theory as limit distributions of eigenvalues for matrices of the form X^*X where X is a rectangular Gaussian matrix, it is the Pastur-Marchenko distribution.

Like in many other situations, relations among probability distributions signal structural connections, in this case a connection between free probability theory and random matrix theory.

Figure 1

Free probability theory can be described as noncommutative probability theory endowed with the definition of free independence. The next sections briefly explain the two terms: noncommutative probability theory and free independence.

1.2 Noncommutative probability theory

In classical probability theory, numerical random variables are measurable functions on a space of events Ω endowed with a probability measure μ , i.e. a positive measure of mass one. The expectation $E(f)$ of a random variable f is the integral $\int f d\mu$.

Roughly speaking, noncommutative probability theory replaces the ring of numerical random variables by a possibly noncommutative algebra \mathcal{A} over \mathbb{C} with unit $1 \in \mathcal{A}$, which is endowed with a linear expectation functional $\varphi : \mathcal{A} \rightarrow \mathbb{C}$, such that $\varphi(1) = 1$. (\mathcal{A}, φ)

is a *noncommutative probability space* and elements $a \in \mathcal{A}$ are *noncommutative random variables*.

Often (\mathcal{A}, φ) is an algebra of bounded operators on a Hilbert space \mathcal{H} and the functional φ is defined by a unit-vector $\xi \in \mathcal{H}$, i.e., $\varphi(a) = \langle a\xi, \xi \rangle$. Typically quantum mechanical quantities $a \in \mathcal{A}$ are described in this way and ξ is the state-vector.

The *distribution of a random variable* $a \in \mathcal{A}$ is the linear map $\mu_a : \mathbb{C}[X] \rightarrow \mathbb{C}$ so that $\mu_a(P) = \varphi(P(a))$. The information encoded in μ_a is the same as giving the collection of moments $(\varphi(a^n))_{n \geq 0}$.

Similarly for a family $\alpha = (a_i)_{i \in I}$ of random variables in \mathcal{A} , the distribution is the map $\mu_\alpha : \mathbb{C}\langle X_i \mid i \in I \rangle \rightarrow \mathbb{C}$ so that $\mu_\alpha(P) = \varphi(P((a_i)_{i \in I}))$, where $\mathbb{C}\langle X_i \mid i \in I \rangle$ is the algebra of noncommutative polynomials in the indeterminates $(X_i)_{i \in I}$. Like in the one-variable case, μ_α contains the same information as the noncommutative moments $\varphi(a_{i_1} a_{i_2} \dots a_{i_p})$.

In the case of a self-adjoint operator $a = a^*$, μ_a can be identified with a compactly supported probability measure on a . Indeed if $E(a; \omega)$ is the projection-valued spectral measure of a , then

$$\varphi(P(a)) = \langle P(a)\xi, \xi \rangle = \int P(t) \langle dE(a; (-\infty, t)\xi, \xi \rangle$$

i.e., μ_a “is” $E(a; \cdot)\xi, \xi$.

The usual context for free entropy theory is the more restricted one of a *tracial W^* -probability space* (M, τ) . This means that M is a W^* -algebra (synonymous to von Neumann algebra) and that the expectation function τ is a trace. This means that M is a self-adjoint algebra of bounded operators on a Hilbert space \mathcal{H} (i.e. $T \in M \Rightarrow T^* \in M$) which is weakly closed (i.e., if for some net $(T_i)_{i \in I}$ in M , we have $\langle T_i h, k \rangle \rightarrow \langle Th, k \rangle$ for all $h, k \in \mathcal{H}$, then $T \in M$). The condition on τ is that $\tau(ST) = \tau(TS)$ for all $S, T \in M$.

If (Ω, Σ, μ) is a probability space then $M = L^\infty(\Omega, \Sigma, \mu)$ acting as multiplication operators on $L^2(\Omega, \Sigma, \mu)$ is a W^* -algebra and the expectation functional τ defined by the vector $1 \in L^2$ is trivially a trace since M is commutative. Note that τ coincides with the classical expectation functional on L^∞ defined by μ . Thus tracial W^* -probability spaces subsume the context of classical probability spaces.

A fundamental class of tracial W^* -probability spaces is generated by discrete groups G . Let λ be the left regular representation of G on $\ell^2(G)$, i.e. $\lambda(g)e_h = e_{gh}$ where e_g , $g \in G$, are the canonical basis vectors in $\ell^2(G)$. Then the *von Neumann algebra* $L(G)$ is defined as the weakly closed linear space of $\lambda(G)$. Roughly speaking, $L(G)$ consists of those left convolution operators $\sum_{g \in G} c_g \lambda(g)$ which are bounded on ℓ^2 . The trace τ is the von Neumann trace which is defined by the basis vector e_e (or any other e_g). Note that $\tau(\sum_g c_g \lambda(g)) = c_e$ (here the next e denotes the neutral element in G).

1.3 Free independence

A family of subalgebras $(\mathcal{A}_i)_{i \in I}$, with $1 \in \mathcal{A}_i$, in (\mathcal{A}, φ) is *freely independent* if

$$\varphi(a_1 \dots a_n) = 0$$

whenever $\varphi(a_j) = 0$, $1 \leq j \leq n$ and $a_j \in \mathcal{A}_{i(j)}$ with $i(j) \neq i(j+1)$, $1 \leq j \leq n-1$. A family of subsets $(\omega_i)_{i \in I}$ in (A, φ) is freely independent if the algebras \mathcal{A}_i generated by $\{1\} \cup \omega_i$ are freely independent.

The above definition means that products of centered variables, such that consecutive ones are in different algebras, have expectation zero. Note that this does not preclude that $i(j) = i(k)$ as long as $|j - k| \geq 2$.

In general free independence requires that variables be very far from commuting. For instance, if X, Y are freely independent and centered $\varphi(X) = \varphi(Y) = 0$, then the free independence condition requires that $\varphi(XYXY) = 0$ while commutation of X and Y would imply

$$\varphi(XYXY) = \varphi(X^2Y^2) = \varphi(X^2)\varphi(Y^2)$$

where the last equality is derived from free independence

$$\varphi((X^2 - \varphi(X^2)1)(Y^2 - \varphi(Y^2)1)) = 0.$$

Thus commutation is impossible if $\varphi(X^2) \neq 0$, $\varphi(Y^2) \neq 0$.

A basic example of free independence is provided by groups. A family $(G_i)_{i \in I}$ of subgroups of a group G is free, in the sense of group theory if there is no non-trivial algebraic relation in G among the G_i 's which translates into the requirement that $g_1 g_2 \dots g_n \neq e$ whenever $g_j \neq e$, $1 \leq j \leq n$ and $g_j \in G_{i(j)}$ with $i(j) \neq i(j+1)$, $1 \leq j \leq n-1$. It can be shown that in $(L(G), \tau)$ the free independence of the sets $(\lambda(G_i))_{i \in I}$ is equivalent to the requirement that the family of subgroups $(G_i)_{i \in I}$ is algebraically free. Note that this is also equivalent to the free independence of the von Neumann algebras generated by the $\lambda(G_i)$.

1.4 Random matrices in the large N limit

The explanation found in [33] for the clues to a connection between free probability and random matrices is that free independence occurs asymptotically among large random matrices.

Very roughly the connection is as follows. A random matrix is a classical matrix-valued random variable. At the same time random matrices give rise to operators, i.e. to noncommutative random variables. Note that the passage from the classical variable to the noncommutative one means forgetting part of the information (the noncommutative moments

can be computed from the classical distribution but not vice versa). Then under certain conditions (like unitary invariance) independent random matrices give rise asymptotically as their size increases to freely independent noncommutative random variables.

The noncommutative probability framework for random matrices is given by the algebras

$$\mathcal{A}_N = L^{-\infty}(X, \mathcal{M}_N)$$

where $(X, \Sigma, d\sigma)$ is a probability space, \mathcal{M}_N denotes the $N \times N$ complex matrices and $L^{-\infty}$ stands for the intersection of L^p -spaces $1 \leq p < \infty$. The expectation functional on \mathcal{A}_N is $\varphi_N : \mathcal{A}_N \rightarrow \mathbb{C}$ given by

$$\varphi_N(T) = N^{-1} \int_X \text{Tr}(T(x)) d\sigma(x) .$$

The simplest instance of asymptotic free independence is provided by a pair of Gaussian matrices. Let $T_j^{(N)} = (a_{p,q;j}^{(N)})_{1 \leq p,q \leq N} \in \mathcal{A}_N$, $j = 1, 2$, where $a_{p,q;j}^{(N)} = a_{q,p;j}^{(N)}$ and $\{a_{p,q;j}^{(N)} \mid 1 \leq p \leq q \leq N, j = 1, 2\}$ are independent $(0, N^{-1})$ -Gaussian. Then $T_1^{(N)}, T_2^{(N)}$ are asymptotically free as $N \rightarrow \infty$, in the sense that the algebraic relations among the noncommutative moments of the pair $(T_1^{(N)}, T_2^{(N)})$ which represent the free independence conditions, are satisfied in the limit $N \rightarrow \infty$.

Among the uses of asymptotic freeness of random matrices are the study of the large N -limit of random matrices with free probability techniques on one hand and on the other hand the operator algebra applications. Operator algebras such as the von Neumann algebras of free groups $L(F(n))$ are generated by free random variables and can therefore be viewed as asymptotically generated by random matrices. This has provided the intuitive background for many new results.

1.5 Free independence with amalgamation

In usual probability theory conditional independence amounts to replacing the scalar expectation functional with the conditional expectation w.r.t. a sub- σ -algebra of events, i.e., the expectation takes values in a sub-algebra of the algebra of random variables.

The free analogue of conditional independence is free independence with amalgamation. The context is a \mathcal{B} -valued probability space, i.e. $(\mathcal{A}, E, \mathcal{B})$ where $1 \in \mathcal{B} \subset \mathcal{A}$ is an inclusion of unital algebras over \mathbb{C} and $E : \mathcal{A} \rightarrow \mathcal{B}$ is \mathcal{B} - \mathcal{B} -bilinear and $E|_{\mathcal{B}} = \text{id}_{\mathcal{B}}$. Then a family of subalgebras $(\mathcal{A}_i)_{i \in I}$, $\mathcal{B} \subset \mathcal{A}_i \subset \mathcal{A}$ is \mathcal{B} -freely independent if $E(a_1 \dots a_n) = 0$ whenever $E(a_j) = 0$, $1 \leq j \leq n$, $a_j \in \mathcal{A}_{i(j)}$, $i(k) \neq i(k+1)$, $i \leq k < n$.

If (M, τ) is a tracial W^* -probability space, with faithful τ (i.e., $\tau(x^*x) = 0 \Rightarrow x = 0$) then there are canonical conditional expectations onto von Neumann subalgebras. If $I \in$

$N \subset M$ is a von Neumann subalgebra, then $\langle m_1, m_2 \rangle = \tau(m_2^* m_1)$ is an inner product on M and E_N is defined as the orthogonal projection of M onto the Hilbert space completion of N . It turns out that actually $E_N(M) \subset N$ and $\|E_N m\| \leq \|m\|$. Of course for the L^2 -norm $\|m\|_2 = (\tau(m^* m))^{\frac{1}{2}}$ we also have $\|E_N m\|_2 \leq \|m\|_2$. Moreover E_N is N - N -bilinear. This is clearly a generalization of the classical situation where $M = L^\infty(\Omega, \Sigma, \mu)$ and $N = L^\infty(\Omega, \Sigma_1, \mu)$ with $\Sigma_1 \subset \Sigma$ a σ -subalgebra.

If G is a group and H a subgroup let $L(H)$ be identified with the W^* -subalgebra generalized by $\lambda(H)$ in $L(G)$. Then

$$E_{L(H)} \sum_{g \in G} c_g \lambda(g) = \sum_{g \in H} c_g \lambda(g) .$$

Also if $H \subset G_i \subset G$ is a family of subgroups indexed by I , then the $L(G_i)$ are $L(H)$ -freely independent in $(L(G), E_{L(H)})$ iff the subgroups G_i are algebraically free with amalgamation over H .

1.6 Background references

The beginning of free probability theory is the paper [31] and the connection to random matrices is in [33]. A comprehensive introduction to free probability theory is given in [42] and for probabilists (i.e. for readers who prefer operator algebras kept to a minimum) there are the St-Flour lectures [39]. Some standard operator algebra books are [7], [8], [19], [29].

2 Matricial Microstates Approach to Free Entropy

2.1 Underlying idea

Shannon's entropy of a continuous n -dimensional distribution ([23]) is given by the formula

$$H(f_1, \dots, f_n) = - \int_{\mathcal{R}^n} p(t_1, \dots, t_n) \log p(t_1, \dots, t_n) dt_1 \dots dt_n$$

where f_1, \dots, f_n are real-valued random variables with Lebesgue absolutely continuous joint distribution with density $p(t_1, \dots, t_n)$. A free analogue to $H(f_1, \dots, f_n)$ will be a number $\chi(X_1, \dots, X_n)$ [35 II] associated to an n -tuple of self-adjoint elements X_j ($1 \leq j \leq n$) in a tracial W^* -probability space (M, τ) , the properties of χ w.r.t. free independence being parallel to those of H w.r.t. classical independence.

Information-theoretic and physical entropy though different concepts, have also much in common. In particular, the formula for $H(f_1, \dots, f_n)$ can be derived from Boltzmann's

fundamental formula $S = k \log W$. The connection to the Boltzmann formula and the fact that free independence occurs asymptotically among large matrices, are the key to the definition of χ .

Boltzmann's formula says that the entropy S of a "macrostate" is proportional to the logarithm of its "Wahrscheinlichkeit" W (probability), where the probability of the "macrostate" is obtained by counting how many "microstates" correspond to that "macrostate". For mathematical purposes, microstates are often associated with a given degree of approximation, and one then takes a normalized limit when the number of microstates goes to infinity, followed by a limit improving the approximation.

For simplicity, here is how this works for the entropy of a discrete random variable with outcomes $\{1, \dots, n\}$ with probabilities p_1, \dots, p_n . The microstates are the set $\{1, \dots, n\}^N = \{f \mid f : \{1, \dots, N\} \rightarrow \{1, \dots, n\}\}$ and the microstates which approximate the discrete distribution are $\Gamma(p_1, \dots, p_n; \varepsilon, N)$ consisting of those f such that

$$\left| \frac{|f^{-1}(j)|}{N} - p_j \right| < \varepsilon$$

($|f^{-1}(j)|$ the number of elements in the pre-image.) One then takes the limit of

$$N^{-1} \log |\Gamma(p_1, \dots, p_n; \varepsilon, N)|$$

as $N \rightarrow \infty$ and then lets ε go to zero. Using repeatedly Stirling's formula one gets the familiar $-\sum p_j \log p_j$ result in the end.

To define χ , the microstates will be matricial.

2.2 The definition of $\chi(X_1, \dots, X_n)$ [35 II]

Given $X_j = X_j^* \in M$, $1 \leq j \leq n$, where (M, τ) is a tracial W^* -probability space, the set of approximating matricial microstates will be denoted $\Gamma_R(X_1, \dots, X_n; m, k, \varepsilon)$ where $R > 0$, $m \in \mathbb{N}$, $k \in \mathbb{N}$, $\varepsilon > 0$. Here R is a cut-off parameter, k the size of matrices and (m, ε) the degree of approximation. With \mathcal{M}_k^{sa} denoting the self-adjoint complex $k \times k$ matrices, the approximating microstates are n -tuples $(A_1, \dots, A_n) \in (\mathcal{M}_k^{sa})^n$ such that

$$|\tau(X_{i_1} \dots X_{i_p}) - k^{-1} \text{Tr}(A_{i_1} \dots A_{i_p})| < \varepsilon$$

for all $1 \leq p \leq m$, $(i_1, \dots, i_p) \in \{1, \dots, n\}^p$ and $\|A_j\| < R$, $1 \leq j \leq m$.

Let vol denote the euclidean volume on $(M_k^{sa})^n$ w.r.t. the Hilbert-Schmidt scalar product

$$\langle (A_1, \dots, A_n), (B_1, \dots, B_n) \rangle = \sum_j \text{Tr } A_j B_j .$$

Taking

$$\limsup_{k \rightarrow \infty} (k^{-2} \log \text{vol} \Gamma_R(X_1, \dots, X_n; m, k, \varepsilon) + \frac{n}{2} \log k)$$

and then

$$\sup_{R>0} \inf_{m \in \mathbb{N}} \inf_{\varepsilon > 0}$$

of the result, we obtain $\chi(X_1, \dots, X_n)$.

Note that the cut-off R has only a minor influence, instead of the sup over R we could have taken a fixed R larger than $\|X_j\|$, $1 \leq j \leq n$.

2.3 Basic properties of $\chi(X_1, \dots, X_n)$

χ .1. **Upper Bound** ([35 II])

$$\chi(X_1, \dots, X_n) \leq 2^{-1} n \log(2\pi e n^{-1} C^2) \text{ where } C^2 = \tau(X_1^2 + \dots + X_n^2).$$

In particular $\chi(X_1, \dots, X_n)$ is either finite or $-\infty$.

χ .2. **Subadditivity** ([35 II])

$$\chi(X_1, \dots, X_{m+n}) \leq \chi(X_1, \dots, X_m) + \chi(X_{m+1}, \dots, X_{m+n}).$$

χ .3. **Semicontinuity** ([35 II])

Assume $\|X_j^{(p)}\| \leq C < \infty$, $1 \leq j \leq n$, $p \in \mathbb{N}$ and $(X_1^{(p)}, \dots, X_n^{(p)})$ converges in distribution to (X_1, \dots, X_n) , i.e.

$$\lim_{p \rightarrow \infty} \tau(X_{i_1}^{(p)} \dots X_{i_k}^{(p)}) = \tau(X_{i_1} \dots X_{i_k})$$

for all noncommutative moments. Then

$$\limsup_{p \rightarrow \infty} \chi(X_1^{(p)}, \dots, X_n^{(p)}) \leq \chi(X_1, \dots, X_n).$$

χ .4. **One Variable Case.** ([35 II])

$$\chi(X) = \iint \log |s - t| d\mu(s) d\mu(t) + \frac{3}{4} + \frac{1}{2} \log 2\pi$$

where μ denotes the distribution of X . Thus, up to constants, $\chi(X)$ is minus the logarithmic energy of μ .

χ .5. **Additivity and Free Independence.** ([35 IV])

Assume $\chi(X_j) > -\infty$, $1 \leq j \leq n$. Then

$\chi(X_1, \dots, X_n) = \chi(X_1) + \dots + \chi(X_n)$ **iff** X_1, \dots, X_n are freely independent.

χ .6. Semicircular Maximum. ([35 II])

Assume $\tau(X_1^2) = \dots = \tau(X_n^2) = 1$. Then $\chi(X_1, \dots, X_n)$ is maximum **iff** X_1, \dots, X_n are freely independent and have (0,1)-semicircular distributions.

χ .7. Infinitesimal Change of Variables. ([35 IV])

Let $\mathbb{C}\langle t_1, \dots, t_n \rangle$ be the ring of noncommutative polynomials in the indeterminates t_1, \dots, t_n endowed with the involution $*$ so that $(ct_{i_1} \dots t_{i_p}) = \bar{c}t_{i_p} \dots t_{i_1}$. Then:

$$\begin{aligned} & \left. \frac{d}{d\varepsilon} \chi(X_1 + \varepsilon P_1(X_1, \dots, X_n), \dots, X_n + \varepsilon P_n(X_1, \dots, X_n)) \right|_{\varepsilon=0} \\ &= \sum_{1 \leq j \leq n} (\tau \otimes \tau)(\partial_j P_j(X_1, \dots, X_n)) \end{aligned}$$

where $P_j = P_j^* \in \mathbb{C}\langle t_1, \dots, t_n \rangle$ and $\partial_j : \mathbb{C}\langle t_1, \dots, t_n \rangle \rightarrow \mathbb{C}\langle t_1, \dots, t_n \rangle \otimes \mathbb{C}\langle t_1, \dots, t_n \rangle$ is given by $\partial_j \cdot t_{i_1} \dots t_{i_p} = \sum_{i_k=j} t_{i_1} \dots t_{i_{k-1}} \otimes t_{i_{k+1}} \dots t_{i_p}$.

χ .8. Degenerate Convexity. ([35 III])

Assume $n \geq 2$ and there are trace-states τ', τ'' on $A = W^*(X_1, \dots, X_n)$ so that $\tau' \neq \tau''$ and $\tau = \theta\tau' + (1 - \theta)\tau''$ on A , $0 < \theta < 1$. Then $\chi((X_1, \dots, X_n)) = -\infty$.

Remarks:

- a) It is an important open problem, whether replacing the lim sup in the definition of χ by a lim inf (as $k \rightarrow \infty$) yields the same quantity. While this is unresolved it is sometimes convenient to use χ_ω , ω an ultrafilter on \mathbb{N} , the quantity obtained by replacing the lim sup by a limit as $k \rightarrow \omega$.
- b) Generalizing the “if”-part of χ .5. to groups of variables runs into the problem discussed in a). There is a partial generalization ([36])

$$\chi_\omega(X_1, \dots, X_{m+n}) = \chi_\omega(X_1, \dots, X_m) + \chi_\omega(X_{m+1}, \dots, X_{m+n})$$

if $\{X_1, \dots, X_m\}$ and $\{X_{m+1}, \dots, X_{m+n}\}$ are freely independent.

- c) We preferred to state the weaker infinitesimal version of the change of variable formula because it is easier to state and will be used later. Roughly the change of variable formula is of the form:

$$\chi(F_1(X_1, \dots, X_n), \dots, F_n(X_1, \dots, X_n)) = \chi(X_1, \dots, X_n) + \log |\det|(DF(X_1, \dots, X_n))$$

where there is a long list of details about the noncommutative power series (F_1, \dots, F_n) , the Kadison-Fuglede determinant $|\det|$ and the differential DF for which the reader is referred to the original paper [35 II].

d) Given X_1, \dots, X_n and $m \in \mathbb{N}$, $\varepsilon > 0$ is there $k \in \mathbb{N}$ and $R > 0$ so that

$$\Gamma_R(X_1, \dots, X_n)m, k, \varepsilon) \neq \emptyset ?$$

This very basic question is equivalent to a problem of A.Connes on embedding II_1 -factors into the ultraproduct of the hyperfinite II_1 -factor.

e) The “if”-part of $\chi.5$. relies essentially on asymptotic freeness of random matrices. What the result and its proof show, is a sharp difference between one- and multi-random matrix theory. Roughly, if $n = 1$, then sets of microstates $\Gamma_R(X; m, k, \varepsilon)$ will be like tubes around the unitary orbit of some microstate $\{UAU^* \mid U \in \mathcal{U}(n)\}$. If X_1, \dots, X_n are freely independent and $n > 1$, then $\Gamma_R(X_1, \dots, X_n; m, k, \varepsilon)$ is much larger than a tube around $\{(UAU^*, \dots, UA_nU^*) \mid U \in \mathcal{U}(n)\}$, actually up to sets, the measure of which goes to 0 as $k \rightarrow \infty$, it is more like the product of tubes around the orbits of the components, i.e. $\{UA_kU^* \mid U \in \mathcal{U}(n)\}$.

2.4 The free entropy dimension [35 II]

The free entropy being a normalized limit of logarithms of volumes of sets of matricial microstates, there is also a corresponding normalized dimension of sets of microstates. The definition is reminiscent of the definition of the Minkowski content.

The *free entropy dimension* $\delta(X_1, \dots, X_n)$ is given by the formula

$$\delta(X_1, \dots, X_n) = n + \limsup_{\varepsilon \downarrow 0} \frac{\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n)}{|\log \varepsilon|}$$

where S_1, \dots, S_n have $(0,1)$ -semicircular distributions and $\{X_1, \dots, X_n\}, \{S_1\}, \dots, \{S_n\}$ are freely independent.

In a number of applications it is necessary for technical reasons to use a modification $\delta_0(X_1, \dots, X_n)$ of δ . It is not known whether δ and δ_0 are actually different. δ_0 is obtained by replacing $\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n)$ in the definition of δ by $\chi(X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n : S_1, \dots, S_n)$ where $\chi(X_1, \dots, X_n : Y_1, \dots, Y_p)$ is defined like χ using

$$\Gamma_R(X_1, \dots, X_n : Y_1, \dots, Y_p; m, k, \varepsilon) = pr_{\{1, \dots, n\}} \Gamma_R(X_1, \dots, X_n, Y_1, \dots, Y_p; m, k, \varepsilon)$$

Since all this becomes rather technical, we will limit our discussion to δ in the rest of this section.

Here are some basic properties of δ .

- a) $\delta(X_1, \dots, X_n) \leq n$. We also have $\delta(X_1, \dots, X_n) \geq 0$ when the problem in 2.3–Remark d) has an affirmative answer for X_1, \dots, X_n .
- b) $\delta(X_1, \dots, X_{m+n}) \leq \delta(X_1, \dots, X_m) + \delta(X_{m+1}, \dots, X_{m+n})$
- c) $\delta(X_1, \dots, X_n) = \delta(X_1) + \dots + \delta(X_n)$ if X_1, \dots, X_n are freely independent.
- d) $\delta(X) = 1 - \sum_{t \in \mathcal{R}} (\mu(\{t\}))^2$ where μ is the distribution of X .
- e) $\chi(X_1, \dots, X_n) > -\infty \Rightarrow \delta(X_1, \dots, X_n) = n$.

2.5 Operator algebra applications

Free entropy has led to new results on von Neumann algebras, in particular the solution of some old problems has been found. The new results are about separable II_1 factors, i.e. von Neumann algebras M of infinite dimension acting on separable Hilbert spaces, which have a faithful trace-state τ and trivial center $Z(M) = \mathbb{C}I$. Typical examples are the $L(G)$'s where G is a countable discrete group with infinite conjugacy classes.

1° Absence of Cartan Subalgebras ([35 III])

The free group factors $L(F(n))$ ($n \geq 2$) have no Cartan subalgebras. A Cartan subalgebra $A \subset M$ (M a II_1 factor) is a maximal abelian W^* -subalgebra, the normalizer of which $N(A) = \{u \in M \mid u \text{ unitary, } uAu^* = A\}$ generates M . The concept mimics the properties of the algebra of diagonal matrices inside the algebra of $n \times n$ matrices. M has a Cartan subalgebra iff it can be obtained from an ergodic measurable equivalence relation via a construction of Feldman and Moore ([13]). It was an open problem whether all separable II_1 factors arise this way from ergodic theory.

2° Prime II_1 factors ([15 II])

$L(F(n))$ ($n \geq 2$) is prime, i.e. is not a W^* -tensor product $M_1 \otimes M_2$ of ∞ -dimensional von Neumann algebras. The existence of separable II_1 factors was also an old open question.

3° Products of abelian subalgebras ([30])

If n is large enough, $L(F(n))$ is not the 2-norm closure of the linear span of a product $A_1 \dots A_m$ of m abelian W^* -subalgebras.

Using a fundamental theorem of A. Connes, by which all separable II_1 -factors $L(G)$ with G amenable are isomorphic, it follows that in the amenable case $L(G) = \overline{\text{span } A_1 A_2}$

for a pair of abelian W^* -subalgebras. This is in sharp contrast with the $L(F(n))$ situation.

The principle underlying the proofs of these results is to show that a certain property (existence of a Cartan subalgebra, non-primeness, product of abelian, etc.) implies that a generator $X_j = X_j^*$ ($1 \leq j \leq n$) of the von Neumann algebra has $\chi(X_1, \dots, X_n) = -\infty$. On the other hand $L(F(n))$ has a generator X_1, \dots, X_n with $\chi(X_1, \dots, X_n) > -\infty$ (consider Borel-logarithms of the generating unitaries $\lambda(g_1) \dots \lambda(g_n)$ and use $\chi.4$ and $\chi.5$). This kind of result, started by the absence of Cartan algebras result ([35 III]) has meant developing increasingly ingenious ways of estimating volumes of matricial microstates for generators ([12],[15],[30]).

Note also that for most of the above results there are stronger forms, where $\chi(X_1, \dots, X_n) = -\infty$ is replaced by $\delta_0(X_1, \dots, X_n) \leq 1$ for a generator. In this direction there is also the following recent result.

4° **Property T** ([15 III])

If $X_j = X_j^*$ ($1 \leq j \leq n$) is a generator of $L(SL(rm + 1; \mathbb{Z}))$ ($m \geq 1$) then $\delta_0(X_1, \dots, X_n) \leq 1$.

The restriction to odd numbers $2m + 1$ is only to insure factoriality (i.e. trivial center).

2.6 Comments on the microstates approach

The use of microstates, per se, in the definition of free entropy, should not bother us too much. There are many other situations in mathematics where huge auxiliary objects are used to define some basic invariants (singular homology may come to mind for instance). On the other hand, the technical difficulties in this approach which prevented us from completing the theory (see, for instance, Remarks a) and b) in 2.3) are a problem.

Much impetus for further developing free entropy theory is provided by von Neumann algebras. There is some hope that with stronger free entropy tools at hand, the currently best known problem in the area may be settled in the affirmative:

$L(F(n))$ isomorphism problem.

Does $L(F(n)) \simeq L(F(m))$ imply $m = n$?

An even more far-fetched question is whether for the free entropy dimension, or for some variant of it, there is an affirmative answer to:

The entropy dimension problem. If $X_j = X_j^* \in M$, $Y_k = Y_k^* \in M$, $1 \leq j \leq n$, $1 \leq k \leq m$, does $W^*(X_1, \dots, X_n) = W^*(Y_1, \dots, Y_m)$ imply $\delta(X_1, \dots, X_n) = \delta(Y_1, \dots, Y_m)$?

Under certain conditions, an affirmative answer to the preceding problem would follow (see [35 II]) from an affirmative answer to:

Semicontinuity of δ problem. If $X_j^{(p)} = X_j^{(p)*} \in M$, $X_j = X_j^* \in M$, $1 \leq j \leq n$, $p \in \mathbb{N}$ are so that $s - \lim_{p \rightarrow \infty} X_j^{(p)} = X_j$ does it follow that $\liminf_{p \rightarrow \infty} \delta(X_1^{(p)}, \dots, X_n^{(p)}) \geq \delta(X_1, \dots, X_n)$?

Little is known about these questions. About the semicontinuity problem it is only known that in the rather uninteresting case $n = 1$, the answer is yes ([35 II]). For certain variants of δ , the much weaker free entropy dimension problem, with the W^* -algebras replaced by the algebras (no closures) of the X 's and Y 's, the answer is affirmative ([36]). Also the isomorphisms of various free product von Neumann algebras ([10],[11],[22],[34]) seem not to contradict the invariance of δ on generators. Finally, it is known [22] that there are only two possibilities in the isomorphism problem: either all $L(F(n+1))$, $n \in \mathbb{N} \cup \{\infty\}$ are isomorphic or all are non-isomorphic.

3 Infinitesimal Approach to Free Entropy

3.1 Fisher information background

The Fisher information $\mathcal{J}(f)$ of a real random variable f is the derivative of the entropy in the direction of a Brownian motion starting at f , or equivalently:

$$\frac{1}{2} \mathcal{J}(f) = \lim_{\varepsilon \downarrow 0} (H(f + \varepsilon^{\frac{1}{2}} g) - H(f))$$

where g is a (0,1)-Gaussian variable independent of f . Using the Brownian motion starting at f one can then express H via \mathcal{J} ,

$$H(g) - H(f) = \frac{1}{2} \int_0^\infty (\mathcal{J}(f + t^{\frac{1}{2}} g) - (1+t)^{-1}) dt .$$

On the other hand, if the distribution of f is Lebesgue absolutely continuous with smooth density p , then one finds

$$\mathcal{J}(f) = \int_{\mathcal{R}} \frac{(p'(t))^2}{p(t)} dt .$$

The last formula can also be expressed as an L^2 -norm

$$\mathcal{J}(f) = \left\| \frac{p'}{p} \right\|_{L^2(\mathcal{R}, pd\lambda)}^2$$

or equivalently

$$\mathcal{J}(f) = E \left(\left(\frac{p'}{p}(f) \right)^2 \right) .$$

The Fisher information initially appeared in statistics, where it was defined by the preceding formula with $\frac{p'}{p}(f)$ being the so-called score-function of f . The score is also fundamental for other reasons: a) infinitesimally the effect on the distributions of the perturbations $f + \varepsilon^{\frac{1}{2}}g$ and $f + \frac{\varepsilon}{2} \frac{p'}{p}(f)$ is the same; b) the score is a gradient for the entropy when the perturbations of f are of the form $f + \varepsilon Q(f)$ where Q is a polynomial.

Related to property a) of the score the element $p'/p \in L^2(\mathcal{R}, pd\lambda)$ can also be described as:

$$\frac{p'}{p} = - \left(\frac{d}{dt} \right)^* 1$$

where $\frac{d}{dt}$ is the operator of derivation densely defined on polynomials in $L^2(\mathcal{R}, pd\lambda)$ and p is smooth with compact support. In particular,

$$\mathcal{J}(f) = \left\| \left(\frac{d}{dt} \right)^* 1 \right\|_{L^2(\mathcal{R}, pd\lambda)}^2$$

Based on properties of the free entropy χ and on one-dimensional computations [35 I], it turns out [35 V] that the free analogue of the Fisher information can be obtained, roughly speaking, by replacing the operator of derivation d/dt by some difference quotient, which sends a polynomial $P(t)$ to the two-variable polynomial:

$$\frac{P(s) - P(t)}{s - t} .$$

Dealing with several noncommuting variables will involve noncommutative generalizations of the difference quotient, like the derivations appearing in the infinitesimal change of variable formula for χ .

3.2 The free difference quotient

Let $X = X^* \in M$ and $1 \in B \subset M$ be a $*$ -subalgebra such that X and B are algebraically free (i.e., no non-trivial algebraic relation between B and X). We denote by $B[X]$ the algebra generated by B and X and consider the linear map:

$$\partial_{X:B} : B[X] \rightarrow B[X] \otimes B[X]$$

so that

$$\partial_{X:B} b_0 X b_1 X \dots b_n = \sum_{1 \leq k \leq n} b_0 X \dots b_{k-1} \otimes b_k X \dots b_n .$$

With the natural $B[X]$ -bimodule structure on $B[X] \otimes B[X]$, the map $\partial_{X:B}$ is a derivation and it is the only one such that $\partial_{X:B}|B = 0$ and $\partial_{X:B}X = 1 \otimes 1$.

Note that the partial derivation appearing in the infinitesimal change of variable formula for χ correspond to taking $B = \mathbb{C}[X_1, \dots, \widehat{X}_j, \dots, X_n]$ and $X = X_j$ (here X_1, \dots, X_n are algebraically free, noncommuting).

$B[X]$ is a linear subspace of $L^2(M, \tau)$ and we shall consider $L^2(B[X], \tau)$ the closure of $B[X]$.

3.3 The conjugate variable $\mathcal{J}(X : B)$ [35 V]

In the context of the preceding section $\partial_{X:B}$ is a densely defined unbounded operator from $L^2(B[X], \tau)$ to $L^2(B[X], \tau) \otimes L^2(B[X], \tau)$. We define $\mathcal{J}(X : B) = \partial_{X:B}^* 1 \otimes 1$ if it exists and call it the conjugate variable to X (w.r.t. B).

Several other names are appropriate for $\mathcal{J}(X : B)$: noncommutative Hilbert transform, free Brownian gradient, free score. All these designations correspond to properties of $\mathcal{J}(X : B)$ which will be described in what follows. In particular the passage from the usual (partial) derivative to the free difference quotient justifies the “free score” name.

Here are some basic facts about $\mathcal{J}(X : B)$.

$\mathcal{J}.1.$ Hilbert transform. If the distribution of X is Lebesgue absolutely continuous and has density $p \in L^3(\mathcal{R}, d\lambda)$, then $\mathcal{J}(X : \mathbb{C}) = g(X)$, where $g = 2\pi H p$, with H denoting the Hilbert transform.

$\mathcal{J}.2.$ Enlarging the scalars. If $1 \in C \subset M$ is a $*$ -subalgebra and C and $B[X]$ are freely independent in (M, τ) then

$$\mathcal{J}(X : B) = \mathcal{J}(X : B \vee C)$$

where $B \vee C$ is the algebra generated by B and C . (There is a strengthening of this in [25]: it suffices to assume C and $B[X]$ are freely independent over B in (M, E_B) .)

J.3. Semicircular perturbations. If S is (0,1) semicircular and $B[X]$ and S are freely independent and $\varepsilon > 0$, then

$$\mathcal{J}(X + \varepsilon S : B)b = \varepsilon^{-1} E_{B[X+\varepsilon S]} S .$$

In particular, $\|\mathcal{J}(X + \varepsilon S : B)\| \leq 2\varepsilon^{-1}$, and the set of selfadjoint X for which $\|\mathcal{J}(X : B)\| < \infty$ is norm-dense in the selfadjoint part of M .

J.4. Closability. If $\|\mathcal{J}(X : B)\|_2 < \infty$ then $\partial_{X:B}^*$ is densely defined and $\partial_{X:B}$ is closable.

J.5. Free Brownian gradient. If S is (0,1) semicircular, $B[X]$ and S freely independent, $\|\mathcal{J}(X : B)\|_2 < \infty$ and $\varepsilon > 0$, then:

$$\begin{aligned} \tau(b_0(X + \frac{\varepsilon}{2} \mathcal{J}(X : B))b_1(X + \frac{\varepsilon}{2} \mathcal{J}(X : B)) \dots b_n) \\ = \tau(b_0(X + \varepsilon^{\frac{1}{2}} S)b_1(X + \varepsilon^{\frac{1}{2}} S) \dots b_n) + O(\varepsilon^2) . \end{aligned}$$

J.6. Gradient of χ . Let $X_j = X_j^* \in M$, $1 \leq j \leq n$ and assume that

$\chi(X_1, \dots, X_n) > -\infty$ and that $\mathcal{J}_k = \mathcal{J}(X_k : \mathbb{C}[X_1, \dots, \widehat{X}_k, \dots, X_n])$, $1 \leq k \leq n$ exist. Then

$$\frac{d}{d\varepsilon} \chi(X_1 + \varepsilon P_1, \dots, X_n + \varepsilon P_n)|_{\varepsilon=0} = \sum_{1 \leq k \leq n} \tau(P_k \mathcal{J}_k)$$

where $P_k = P_k^* \in \mathbb{C}[X_1, \dots, X_n]$, $1 \leq k \leq n$.

3.4 $\Phi^*(X_1, \dots, X_n : B)$ ([35 V])

In the infinitesimal approach, the relative Fisher information $\Phi^*(X_1, \dots, X_n : B)$ of an n -tuple of selfadjoint variables X_1, \dots, X_n with respect to the subalgebra B is defined by

$$\Phi^*(X_1, \dots, X_n : B) = \sum_{1 \leq k \leq n} \|\mathcal{J}(X_k : B[X_1, \dots, \widehat{X}_k, \dots, X_n])\|_2^2$$

if the right-hand side is defined and $+\infty$ otherwise. The asterisk is to distinguish quantities in this approach from the corresponding quantities in the matricial microstates approach.

Here are some properties of Φ^* .

$\Phi^*.1.$ Superadditivity.

$$\Phi^*(X_1, \dots, X_n, Y_1, \dots, Y_m : B) \geq \Phi^*(X_1, \dots, X_n : B) + \Phi^*(Y_1, \dots, Y_m : B)$$

$\Phi^*.2.$ Free additivity. If $B[X_1, \dots, X_n]$ and $C[Y_1, \dots, Y_m]$ are freely independent, then

$$\Phi^*(X_1, \dots, X_n, Y_1, \dots, Y_m : B \vee C) = \Phi^*(X_1, \dots, X_n : B) + \Phi^*(Y_1, \dots, Y_m : C).$$

$\Phi^*.3.$ Free Cramer-Rao inequality. $\Phi^*(X_1, \dots, X_n : B)\tau(X_1^2 + \dots + X_n^2) \geq n^2$. Equality holds iff X_j are semicircular with $\tau(X_j) = 0$ ($1 \leq j \leq n$) and $B, \{X_1\}, \dots, \{X_n\}$ are freely independent.

$\Phi^*.4.$ Free Stam inequality. If $B[X_1, \dots, X_n]$ and $C[Y_1, \dots, Y_m]$ are freely independent, then

$$(\Phi^*(X_1 + Y_1, \dots, X_n + Y_n : B \vee C))^{-1} \geq (\Phi^*(X_1, \dots, X_n : B))^{-1} + (\Phi^*(Y_1, \dots, Y_m : C))^{-1}.$$

$\Phi^*.5.$ Semicontinuity. If $X_j^{(k)} = X_j^{(k)*} \in M$ and $s - \lim_{k \rightarrow \infty} X_j^{(k)} = X_j$, then

$$\liminf_{k \rightarrow \infty} \Phi^*(X_1^{(k)}, \dots, X_n^{(k)} : B) \geq \Phi^*(X_1, \dots, X_n : B).$$

$\Phi^*.6.$ If $\Phi^*(X_1, \dots, X_n : B) = \Phi^*(X_1, \dots, X_n : \mathbb{C}) < \infty$ then $\{X_1, \dots, X_n\}$ and B are freely independent. If $\Phi^*(X_1, \dots, X_n, Y_1, \dots, Y_m : \mathbb{C}) = \Phi^*(X_1, \dots, X_n : \mathbb{C}) + \Phi^*(Y_1, \dots, Y_m : \mathbb{C})$ then $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_m\}$ are freely independent.

3.5 $\chi^*(X_1, \dots, X_n : B)$

The free entropy of X_1, \dots, X_n relative B , in the infinitesimal approach is defined by

$$\chi^*(X_1, \dots, X_n : B) = \frac{1}{2} \int_0^\infty \left(\frac{n}{1+t} - \Phi^*(X_1 + t^{\frac{1}{2}} S_1, \dots, X_n + t^{\frac{1}{2}} S_n : B) \right) dt + \frac{n}{2} \log 2\pi e$$

where the S_j 's are (0,1)-semicircular and $B[X_1, \dots, X_n], \{S_1\}, \dots, \{S_n\}$ are freely independent.

Here are some properties of χ^* .

$$\chi^*.1. \quad \chi(X : \mathbb{C}) = \chi(X).$$

$$\chi^*.2. \quad \chi^*(X_1, \dots, X_n) \leq \frac{n}{2} \log(2\pi n^{-1} C^2) \quad \text{where} \quad C^2 = \tau(X_1^2 + \dots + X_n^2).$$

$$\chi^*.3. \quad \text{If } B[X_1, \dots, X_n] \text{ and } C \text{ are freely independent, then}$$

$$\chi^*(X_1, \dots, X_n : B) = \chi^*(X_1, \dots, X_n : B \vee C).$$

$\chi^*.4$. **Subadditivity.**

$$\chi^*(X_1, \dots, X_n, Y_1, \dots, Y_m : B \vee C) \leq \chi^*(X_1, \dots, X_n : B) + \chi^*(Y_1, \dots, Y_m : C)$$

$\chi^*.5$. **Free additivity.** If $B[X_1, \dots, X_n]$ and $C[Y_1, \dots, Y_m]$ are freely independent then the inequality $\chi^*.4$ is an equality.

$\chi^*.6$. **Semicontinuity.** If $s - \lim_{k \rightarrow \infty} X_j^{(k)} = X_j$ then

$$\limsup_{k \rightarrow \infty} \chi^*(X_1^{(k)}, \dots, X_n^{(k)} : B) \leq \chi^*(X_1, \dots, X_n : B).$$

$\chi^*.7$. **Information log-Sobolev inequality.** If $\Phi^*(X_1, \dots, X_n : B) < \infty$ then

$$\chi^*(X_1, \dots, X_n : B) \geq \frac{n}{2} \log \left(\frac{2\pi n e}{\Phi^*(X_1, \dots, X_n : B)} \right), \text{ in particular}$$

$$\chi^*(X_1, \dots, X_n : B) > -\infty.$$

3.6 Mutual free information and the derivation $\delta_{A:B}$ [35 VI]

In the classical context, if f, g is a pair of numerical random variables with $H(f), H(g), H(f, g)$ finite, then their *mutual information* is

$$I(f; g) = H(f) + H(g) - H(f, g).$$

Via an approximation procedure, the definition of $I(f, g)$ can be extended well beyond the case of finite entropies (even Lebesgue absolute continuity of distributions is not a requirement, see [6]). It also turns out that $I(f, g)$ depends only on the position of the von Neumann algebras of f and g inside the von Neumann algebra of $\{f, g\}$ endowed with the expectation functional [in classical terms: the triple of σ -algebras of f , respectively g , and respectively (f, g) -measureable events and the probability measure]. Note however that there is no infinitesimal theory for $I(f, g)$ unless one is in the finite entropy case and uses the infinitesimal theory for entropy, i.e., there is no infinitesimal theory at the level of algebras, since there is no natural deformation of the pair of algebras in sight.

In the free context, the situation is different. Given two von Neumann subalgebras $1 \in A$, $1 \in B$ in (M, τ) there is a natural “liberation process” which deforms the pair (A, B) to a freely independent pair: $A, U(t)BU(t)^*$ where $\{U(t)\}_{t \geq 0}$ is a multiplicative unitary free Brownian motion which is freely independent from $A \vee B$. This means $\{U(t)\}_{t \geq 0}$ is the free analogue of the corresponding classical Brownian motion on the unit circle and can also be described, in view of the asymptotic freeness of random matrices as the large N limit of Brownian motions on the unitary groups $U(N)$ (see [2]). Via some heuristic considerations

this leads to an infinitesimal approach to a quantity $i^*(A, B)$ which should play the role of the mutual free information for the pair (A, B) .

The infinitesimal approach relies on a derivation

$$\delta_{A:B} : A \vee B \rightarrow (A \vee B) \otimes (A \vee B)$$

which exists under the assumption that A and B are algebraically free (i.e., no non-trivial algebraic relation). Here $(A \vee B) \otimes (A \vee B)$ is with the obvious $A \vee B$ bimodule structure and

$$\begin{aligned} \delta_{A:B}a &= a \otimes 1 - 1 \otimes a & \text{if } a \in A \\ \delta_{A:B}b &= 0 & \text{if } b \in B. \end{aligned}$$

Like in the infinitesimal approach to free entropy, the key construction is the liberation gradient

$$j(A : B) = \delta_{A:B}^* 1 \otimes 1$$

where $\delta_{A:B}$ is viewed as an unbounded operator densely defined on $L^2(W^*(A \vee B), \tau)$ with values in $L^2(W^*((A \vee B) \otimes (A \vee B)), \tau \otimes \tau)$.

We list some of the main properties of $j(A : B)$.

j.1. Liberation gradient. $j(A : B) = -j(A : B)^*$ and

$$\tau \left(\prod_{1 \leq k \leq n}^{\rightarrow} U(\varepsilon) a_k U(\varepsilon)^* b_k \right) = \tau \left(\prod_{1 \leq k \leq n}^{\rightarrow} \exp \left(\frac{\varepsilon}{2} j(A : B) \right) a_k \exp \left(- \frac{\varepsilon}{2} j(A : B) \right) b_k \right) + O(\varepsilon^2)$$

where $a_k \in A$, $b_k \in B$, $\prod_{1 \leq k \leq n}^{\rightarrow}$ denotes the ordered product and $(U(t))_{t \geq 0}$ is the multiplicative unitary free Brownian motion free w.r.t. $A \vee B$.

j.2. $j(A : \mathbb{C}) = 0$ and $\sum_{1 \leq k \leq n} j(A_k : A_1 \vee \dots \vee A_{k-1} \vee A_{k+1} \vee \dots \vee A_n) = 0$

j.3. If A, B, C is freely Markovian (i.e. A and C are freely independent over B in (M, E_B)) then

$$\begin{aligned} j(A : B) &= j(A : B \vee C) \\ j(A : C) &= E_{A \vee C} j(B : C) \end{aligned}$$

j.4. If U is unitary and $A \vee B$ and $\{U, U^*\}$ are freely independent, then

$$j(A : UBU^*) = E_{A \vee UBU^*} j(A : B)$$

and if the distribution of U is absolutely continuous w.r.t. Haar measure, $d\mu = p d\theta$, $p \in L^3$, then

$$j(A : UBU^*) = -iE_{A \vee UBU^*} g(U)$$

where $g(e^{i\theta_1}) = -\frac{1}{2\pi} \text{p.v.} \int \frac{p(e^{i(\theta_1-\theta)})}{\tan(\theta/2)} d\theta$ is the Hilbert transform.

$$\mathbf{j.5.} \quad \|(E_A - E_{\mathbb{C}1})(E_B - E_{\mathbb{C}1})\| \leq \frac{\|j(A : B)\|}{(1 + \|j(A : B)\|^2)^{\frac{1}{2}}}$$

(The left-hand side is the norm of an operator on $L^2(M, \tau)$.)

$$\mathbf{j.6.} \quad j(\mathbb{C}[X_1, \dots, X_n] : B) = \sum_k [\mathcal{J}(X_k : B[X_1, \dots, \hat{X}_k, \dots, X_n]), X_k]$$

(if the right-hand side exists).

$$\mathbf{j.7.} \quad j(A : B) = 0 \Leftrightarrow A, B \text{ are freely independent.}$$

The *liberation Fisher information* φ^* is defined by

$$\varphi^*(A : B) = |j(A : B)|_2^2$$

if $j(A : B)$ exists and $= +\infty$ otherwise.

Among its properties is an inequality for freely Markovian triples A, B, C which resembles the Stam inequality

$$\varphi^*(A : C)^{-1} \geq \varphi^*(A : B)^{-1} + \varphi^*(B : C)^{-1}.$$

Finally, the *mutual free information* i^* is then given by

$$i^*(A : B) = \frac{1}{2} \int_0^\infty \varphi^*(U(t)AU(t)^* : B) dt$$

where $(U(t))_{t \geq 0}$ is the unitary free Brownian motion which is free w.r.t. $A \vee B$.

3.7 A variational problem for $\chi(X_1, \dots, X_n)$

It is a natural variational problem for the free entropy to maximize

$$\chi(X_1, \dots, X_n) - \tau(P(X_1, \dots, X_n)) \tag{*}$$

where $X_j = X_j^* \in (M, \tau)$, $1 \leq j \leq n$ and $P = P^* \in \mathbb{C}\langle t_1, \dots, t_n \rangle$ (see $\chi.7$ in 2.3 for this notation). The question is to find the joint distribution of (X_1, \dots, X_n) for which (*) is maximum. ((M, τ) is a “universal” II_1 factor containing all separable II_1 factors.)

It is interesting to note that this problem, about which we know very little in this generality, appears to be connected to an important class of random matrix models, about which similarly very little is known in full generality. To explain this, we shall consider the critical point condition, which is a consequence of (X_1, \dots, X_n) being a point where the maximum is attained:

$$\begin{aligned} \frac{d}{d\varepsilon} \chi(X_1 + \varepsilon P_1(X_1, \dots, X_n), \dots, X_n + \varepsilon P_n(X_1, \dots, X_n))|_{\varepsilon=0} \\ = \frac{d}{d\varepsilon} \tau(P(X_1 + \varepsilon P_1(X_1, \dots, X_n), \dots, X_n + \varepsilon P_n(X_1, \dots, X_n))|_{\varepsilon=0} \end{aligned}$$

Let ∂_j denote $\partial_{X_j: \mathbb{C}[X_1, \dots, \hat{X}_j, \dots, X_n]}$ and let d_j denote the cyclic derivative w.r.t. X_j , i.e.,

$$d_j = m \circ \sim \circ \partial_j$$

where \sim is the flip for $\mathbb{C}[X_1, \dots, X_n] \otimes \mathbb{C}[X_1, \dots, X_n]$ and

$$m : \mathbb{C}[X_1, \dots, X_n] \otimes \mathbb{C}[X_1, \dots, X_n] \rightarrow \mathbb{C}[X_1, \dots, X_n]$$

is multiplication. Then the critical point condition in view of $\chi.7$ becomes

$$\sum_{1 \leq j \leq n} (\tau \otimes \tau)(\partial_j P_j) = \sum_{1 \leq j \leq n} \tau((d_j P) P_j)$$

which in view of 3.3 means precisely that the conjugate variables

$$\mathcal{J}_k = \mathcal{J}(X_k : \mathbb{C}[X_1, \dots, \hat{X}_k, \dots, X_n])$$

exist and that

$$\mathcal{J}_k = d_k P \quad 1 \leq k \leq n. \quad (**)$$

Note that an equivalent way of stating these conditions is

$$\sum_{i_j=k} \tau(X_{i_1} \dots X_{i_{j-1}}) \tau(X_{i_{j+1}} \dots X_{i_p}) = \tau(X_{i_1} \dots X_{i_p} (d_k P)(X_1, \dots, X_n)) \quad (***)$$

for all $1 \leq k \leq n$ and monomials $X_{i_1} \dots X_{i_p}$.

The same equations (see [9] for instance) appear in the study of the large N limit of the general random multi-matrix model arising from a probability measure with density

$$c_N e^{-N \text{Tr } P(A_1, \dots, A_n)}$$

on the space of n -tuples of hermitian $N \times N$ matrices.

Like in the study of random matrix models also for the variational problem (*), it is natural to assume certain lower bounds for P . For instance the condition

$$\tau(P(X_1, \dots, X_n)) \geq A + B \log \tau(X_1^2 + \dots + X_n^2)$$

where $B > \frac{1}{2}$ combined with $\chi.1$ gives

$$\chi(X_1, \dots, X_n) - \tau(P(X_1, \dots, X_n)) \leq K - \varepsilon \log \tau(X_1^2 + \dots + X_n^2)$$

for some constants K and $\varepsilon > 0$ which then will give a bound on $\tau(X_1^2 + \dots + X_n^2)$ for a maximum.

For the reader familiar with one random matrix models, let us point out that for $n = 1$, the variational problem (*) with $\mu = \mu_X$ the distribution of $X = X_1$, becomes in view of $\chi.4$:

$$\iint \log |s - t| d\mu(s) d\mu(t) - \int P(t) d\mu(t)$$

while the equation (**) in view of $\mathcal{J}.1$ becomes

$$(2\pi H\mu)(X) = P'(X)$$

or equivalently

$$2\pi H\mu(t) = P'(X)$$

μ - almost everywhere (i.e., under continuity conditions for $t \in \text{supp } \mu$).

These are familiar objects in the study of 1-random matrix models in the large N limit and free entropy appears to provide the generalization of these for multi-matrix models.

3.8 Comments

In this section we briefly discuss some of the problems encountered in the effort to complete the theory and we also briefly mention further work in this area, not covered in the previous sections.

Unification problem. The ultimate goal of a complete theory also would mean unification of the matricial microstates approach, the infinitesimal approach and the mutual free information defined using $\delta_{A:B}$ and the liberation process. This would mean in particular proving general results of the form

$$\chi(X_1, \dots, X_n) = \chi^*(X_1, \dots, X_n)$$

and

$$\begin{aligned} i^*(W^*(X_1, \dots, X_n) : W^*(Y_1, \dots, Y_m)) \\ = \chi(X_1, \dots, X_n) + \chi(Y_1, \dots, Y_m) - \chi(X_1, \dots, X_n, Y_1, \dots, Y_m) \end{aligned}$$

(when the χ 's in the right-hand side are finite).

Clearly such results are a long way to go from where the theory is at present. As always skeptics would raise the perspective of a negative answer. On the other hand the results paralleling the classical theory, obtained thus far, coupled with our general faith in beautiful mathematical theories should be reasons for optimism that some form of a complete theory and unification are possible. From a more pedestrian point of view it is clear that unification will also very much depend on solving the technical problems in completing each of the three directions.

Technical problems. Developing free entropy theory in the infinitesimal approach, the problems one is facing at present are “free analysis” questions. Here is perhaps the simplest continuity question one would like to settle in the affirmative:

*is $\mathcal{J}(X_1 + tS_1, \dots, X_n + tS_n) \in L^2(M, \tau)$ a continuous function of $t \in (0, \infty)$?
Here S_1, \dots, S_n are $(0, 1)$ -semicircular on $\{S_1\}, \dots, \{S_n\}$, $\{X_1, \dots, X_n\}$ are freely independent in (M, τ) .)*

The question is equivalent to the apparently weaker question:

is $\Phi^(X_1 + tS_1, \dots, X_n + tS_n)$ as a function of $t \in (0, \infty)$ right continuous? It is known the function is left continuous and decreasing.*

Under this form the one-variable case, $n = 1$, has been answered in the affirmative in [41].

The problem of establishing a change of variables results for $\chi^*(X_1, \dots, X_n)$ also runs into difficulties, part of which are related to continuity questions like the preceding one.

Free Fisher information relative to a completely positive map. Several results in the infinitesimal approach have been shown to hold in a more general framework involving a unital completely positive map $\eta : B \rightarrow B$ ([25]). Instead of letting $\partial_{X:B}$ take values in $M \otimes M$ endowed with the scalar product derived from $\tau \otimes \tau$ one uses the scalar product

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \tau(x_2^* \eta(E_B(x_1^* y_1)) y_2) .$$

One context where this generalization has a natural microstates counterpart occurs in the study of Gaussian random band matrices [24],[17]. Another context involves measure-preserving equivalence relations, and a free probability interpretation [26] of the recent work on the cost of such equivalence relations [14].

Large deviations. Recent work on large deviations of Gaussian random matrices, up to technical differences, can be viewed as aiming to prove a strengthening of the equality of the free entropy via microstates χ to the free entropy χ^* defined via an infinitesimal approach, i.e., a strengthening of the unification problem. Slightly more precisely, the asymptotic of $k^{-2} \log \text{vol } \Gamma(k)$ where $\Gamma(k)$ is a set of matricial microstates specified by giving intervals for a finite number of normalized noncommutative moments, should be evaluated by the supremum of a rate function, involving the free entropy χ^* , over the n -tuples of hermitian operators in tracial W^* -probability spaces satisfying the moment conditions. Even more precisely, the preceding should be amended by taking Gaussian measure, removing cutoffs, replacing usual moments by traces of products of some noncommutative resolvents, etc.

In the one-variable case, both free entropy [35 II] and the large deviation question [1] are completely clarified and fit quite well together. In several variables a complete large deviations result, up to some technical differences on microstates, would imply affirmative answers to the \limsup , versus \liminf problem in Remark a) of 2.2 and of the Connes problem in Remark d) of 2.2. Having in mind that a full large deviations would imply the solution of these difficult problems, note that the n -variable results in [5] provide at present the closest result to a majorization of χ by χ^* . Besides the technical differences concerning microstates pointed out above, there is one more important modification in [5] to be pointed out: Φ^* is modified by the L^2 -distance of $(\mathcal{J}(X_k : \mathbb{C}[X_1, \dots, \hat{X}_k, \dots, X_n]))_{1 \leq k \leq n}$ to the set of cyclic gradients. This leads naturally to the problem whether this L^2 -distance is zero, i.e., whether the modification of Φ^* is not really a modification of the quantity? Very little is known about this. A purely algebraic result in [38] implies the distance is zero when the partial free Brownian gradients $(\mathcal{J}(X_k : \mathbb{C}[X_1, \dots, \hat{X}_k, \dots, X_n]))$ are noncommutative polynomials in X_1, \dots, X_n . In a forthcoming paper by T. Cabanal-Duvillard and A. Guionnet it is shown that the n -tuples of noncommutative random variables for which Connes' problem has an affirmative answer, are in the closure in distribution of those for which the above question has an affirmative answer.

In another direction it is important to note that the large deviation work [5] has brought powerful stochastic analysis techniques, applied to matricial Brownian motions, to bear on the problems in this area.

Some extremal problems. Important classes of operators in II_1 -factors, like the circular elements, are the solution to extremal problems for entropy [21].

The coalgebra of $\partial_{X:B}$. The derivation of $\partial_{X:B}$ is a comultiplication for a coalgebra structure on $B[X]$. This leads to a class of coalgebras where the comultiplication is a derivation, which has remarkable duality properties closely related to results on conjugate

variables $\mathcal{J}(X : B)$ ([31]).

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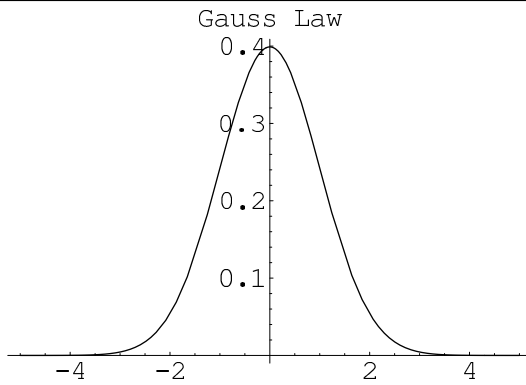
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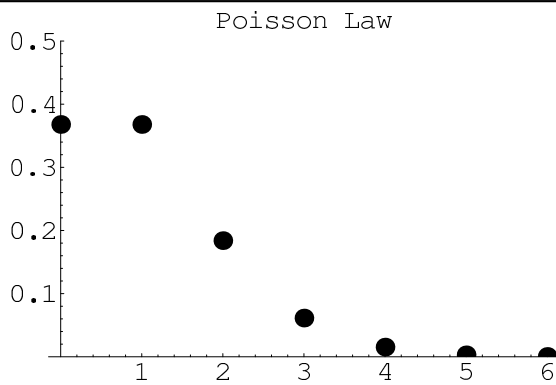
Basic Laws

Continuous lines = Densities, dots = Dirac Measures.

Classical Probability

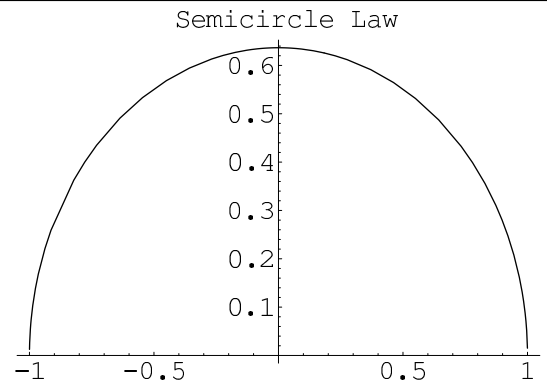


$$c e^{-a x^2} dx$$

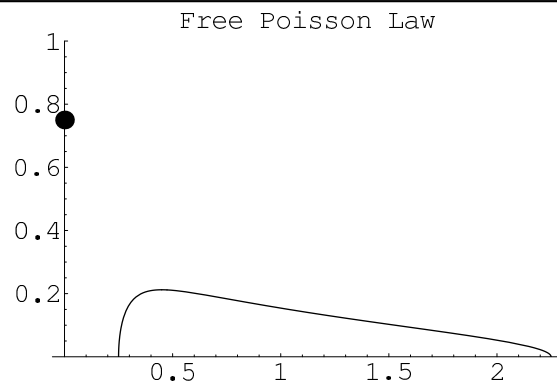


$$\sum_{n \geq 0} e^{-a} \frac{a^n}{n!} \delta_n$$

Free Probability



$$c \sqrt{R^2 - x^2} dx, -R \leq x \leq R$$

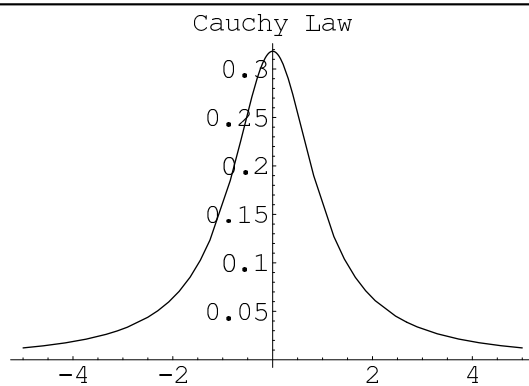


$$(1-a) \delta_0 + \nu, \text{ if } 0 \leq a \leq 1$$

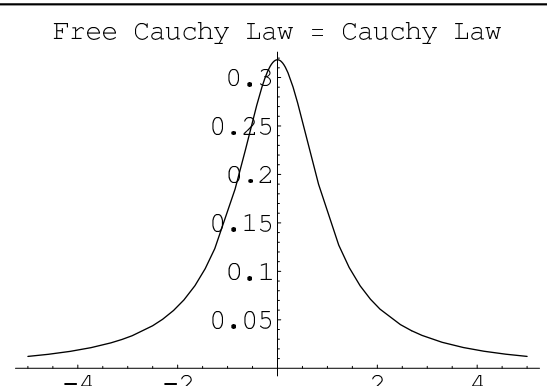
$$\nu, \text{ if } a > 1$$

$$d\nu = (2\pi x)^{-1} (4a - (x - (1+a))^2)^{1/2}$$

$$(1-a^{1/2})^2 \leq x \leq (1+a^{1/2})^2$$



$$c \frac{1}{x^2 + a^2}$$



$$c \frac{1}{x^2 + a^2}$$

